[13.31] Let ***V*** be a vector space and ***T*** be a linear transformation on ***V*** with distinct eigenvalues **1, …, **m, where *m* ≤ n. We furthermore assume

1. For each *j* of multiplicity  ≥ 2 (if any), there are  independent eigenvectors.

Prove there is a basis for ***V*** composed of eigenvectors.

Note: I have reworked Beckmann’s proof to help my understanding of it. I have filled in some details and changed the way he did a few things.

**Solution**. Let  be the multiplicity of eigenvalue ***j*. Since there are *n* eigenvalues, we have that 

If *m* = 1, all the eigenvectors arise from a single eigenvalue and are thus independent by condition (a). From the definition of “basis”, those eigenvectors constitute a basis for ***V***, and we are done. So, we assume *m* > 1. Without loss of generality, if there is a zero eigenvalue, we label it **1.

Let **B** *j* =  be the set of  independent eigenvectors corresponding to *j*. We wish to prove 

comprises a basis for ***V***. Since **B**contains *n* vectors, it suffices to show that these vectors are linearly independent. So, assume



We will be done if we can show that  Set  Since   So, 



Since *m* > 1,  and so we can solve (2) for *Em*:





 (because we are given that the ***j*’s are distinct). We rewrite (4) as





If *m* = 2, then E1 = 0. Plugging this in to (1) yields E2 = 0. Condition (a) implies that  for all *i*, which is what we are trying to prove.

If *m* >2, we continue this process. Since  we can solve (2’) for *am–*1 *Em‑1*:



 or





*bj* ≠ 0 (since  ≠ 0 and ***j* ≠ ***m*), so we next rewrite (4’) as





Continuing …



…



Thus *E*1 = 0.

Plugging *E*1 = 0 into (1*m*–2) yields *E*2 = 0.

Continuing, we get 

From Condition (a), which is what we are trying to prove. 